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Formation of singularities of solutions to the three-dimensional non-relativistic radiation hydrodynamic equations

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ABSTRACT

Classic solutions to the three-dimensional non-relativistic radiation hydrodynamic equations are considered. The formation of singularities in smooth solutions is studied. It is proved that some C^1 solutions, regardless of the size of the initial disturbance, will develop singularities in finite time provided that the initial disturbance satisfies certain conditions.

1. Introduction

The importance of thermal radiation in physical problems increases as the temperature is raised. At moderate temperatures (say, thousands of degrees Kelvin), the role of the radiation is primarily transporting energy by radiative processes. At higher temperatures (say, millions of degrees Kelvin), the energy and momentum densities of the radiation field may become comparable to or even dominate the corresponding fluid quantities (cf. [1]). In this case, the radiation field significantly affects the dynamics of the fluid. This gives rise to the theory of radiation hydrodynamics, which is mainly concerned with the propagation of thermal radiation through a fluid or gas, and the effect of this radiation on the dynamics, see, for example, [2–4], and the references cited therein. The theory of radiation hydrodynamics finds a very broad range of applications, including such diverse astrophysical phenomena as waves and oscillations in stellar atmospheres and envelopes, nonlinear stellar pulsation, supernova explosions, stellar winds, and so on (cf. [5]). It has also direct applications in other areas, for example, the physics of laser fusion, the reentry of vehicles [6], the high temperature combustion phenomena [7], and many others. The mathematical equations governing the radiation hydrodynamics are the Euler equations of compressible fluids coupled with the Boltzmann equation of particle transport.

In this paper we are interested in the finite-time formation of singularities in the three-dimensional non-relativistic radiation hydrodynamic equations. We use $I(x,t,\nu,\Omega)$ to denote the specific intensity of radiation (at time t) at space point $x \in \mathbb{R}^3$, with frequency ν in a direction $\Omega \in S^2$ (the unit sphere of $\mathbb{R}^3$), then the three-dimensional non-relativistic radiation hydrodynamic equations (cf. [8,9]) consist of the following Boltzmann type equation:

$$\begin{aligned}&\frac{1}{c}\frac{\partial I(\nu,\Omega)}{\partial t} + \Omega\cdot\nabla I(\nu,\Omega)\\ &\quad = S(\nu) - \sigma_a(\nu) I(\nu,\Omega)\end{aligned}$$

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$$+\int_0^\infty d\nu' \int_{S^2} \left(\frac{\nu}{\nu'}\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)I(\nu',\Omega') - \sigma_s(\nu\to\nu',\Omega\cdot\Omega')I(\nu,\Omega)\right) d\Omega', \tag{1.1}$$

and the Euler equations:

$$\begin{cases} \rho_t + \nabla\cdot(\rho u) = 0, \\ \left(\rho u + \dfrac{1}{c^2}F_r\right)_t + \nabla P_m + \nabla\cdot(\rho u\otimes u + P_r) = 0, \\ \left(\dfrac{1}{2}\rho u^2 + E_m + E_r\right)_t + \nabla\cdot\left(\left(\dfrac{1}{2}\rho u^2 + E_m + P_m\right)u + F_r\right) = 0, \end{cases} \tag{1.2}$$

where c is the light speed, $S(\nu) = S(x,t,\nu,\Omega,\rho,\theta)$ is the rate of energy emission due to spontaneous processes and $\sigma_a(\nu) = \sigma_a(x,t,\nu,\Omega,\rho,\theta)$ is the absorption coefficient. Similar to absorption, a photon can undergo scattering interactions with matter, and the scattering interaction serves to change the photon's characteristics ν' and Ω' to a new set of characteristics ν and Ω, this leads to the definition of the "differential scattering coefficient" $\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega) = \sigma_s(x,t,\nu'\to\nu,\Omega'\cdot\Omega,\rho,\theta)$. In the Euler equations (1.2), $\rho = \rho(x,t)$ is the density, $u = u(x,t)\in\mathbb{R}^3$ is the velocity of fluid, In non-relativistic radiation hydrodynamic equations, we assume $|u| \ll c$ (cf. [9]), $\theta = \theta(x,t)$ is the temperature, $E_m(\rho,\theta) = \rho e(\theta)$ is the energy, e is the internal energy, $P_m = P_m(\rho,\theta)$ is the pressure, E_r, F_r and P_r represent the radiative energy density, radiative flux and the radiative pressure tensor respectively defined by

$$\begin{cases} E_r = \dfrac{1}{c}\displaystyle\int_0^\infty d\nu \int_{S^2} I(\nu,\Omega)d\Omega, \\ F_r = \displaystyle\int_0^\infty d\nu \int_{S^2} \Omega I(\nu,\Omega)d\Omega, \\ P_r = \dfrac{1}{c}\displaystyle\int_0^\infty d\nu \int_{S^2} \Omega\otimes\Omega I(\nu,\Omega)d\Omega. \end{cases} \tag{1.3}$$

The formation of shock waves is a fundamental physical phenomenon manifested in solutions of the Euler equations for compressible fluids and other related equations, which are a prototypical example of hyperbolic system of conservation laws. This phenomenon can be explained by mathematical analysis by showing the finite-time formation of singularities in the solutions. For the one-dimensional Euler equations, one can use the method of characteristics (cf. [10–16]). For systems with multi-dimensional space variables, the method of characteristics has not been proved tractable. An efficient method, involving the use of averaged quantities, was developed in [17] for hyperbolic systems of conservation laws and was further refined in [18] for the three-dimensional Euler equations. See also [19,20] for the two-dimensional Euler equation and magnetohydrodynamics, [21–23] for the Euler–Poisson equations for spherically, [24] for the Euler–Maxwell equations for spherically symmetric plasma flows, and [25,26] for the multi-dimensional systems of conservation laws. There are few mathematical results on the general radiation hydrodynamical system (1.1)–(1.3) because of their complex structure. Recently, Jiang-Zhong and Jiang-Wang in [27,28] have obtained the local existence of C^1 solutions for the Cauchy problems of the non-relativistic radiation hydrodynamics system (1.1)–(1.3), and showed the finite-time blowup of C^1 solutions under the assumption that the initial data is large, especially, the initial flow velocity must be supersonic in some region (cf. [28,17]). In this paper we study the finite-time formation of singularities for (1.1)–(1.3) and prove that the C^1 solutions must be blown up in finite time without any condition of largeness on the initial data such as the assumptions in [28,17]. The idea of the proof of the main result is motivated by [18]. We will follow [18] closely to adapt the proof in the context of radiation hydrodynamics. In addition, we need to overcome the complex structure of the system, especially from the radiation governed by the Boltzmann equation (1.1) and the radiation terms in the Euler equations (1.2). We use the classic characteristic method to establish a useful estimate between I and I_0, and then use the estimate combined with the property of non-relativistic radiation hydrodynamic equations $|u| \ll c$ to deal with this problem, which requires some new techniques and exact estimates in the proof.

The rest of this paper is organized as follows. In Section 2, we reformulate the problem and state the main result. In Section 3, we prove the main result of this paper.

2. Reformulation and main result

Specifically, we consider the manifestation in the equation of transfer of the quantum statistics (i.e. (1.1)) obeyed by photons. Since photons are bosons, both the processes of emissions and scatterings are enhanced by the number of photons already in the final state following the interaction. This enhancement is generally referred to as resulting from "induced processes" (see, e.g. [9]). The quantitative statement of this enhancement is simply stated as: if Z represents the basic probability of a photon event (emission or scattering, i.e $S(\nu)$ or $\sigma_s(\nu)$) then, due to induced effects, the actual probability Z' is given by (see, e.g. [29]):

$$Z' = Z(1+n),$$

where n is the number of photons in the final state of the transition. In "induced processes" case, (see, e.g. [30])

$$n = \frac{c^2}{2h\nu^3} I(\nu, \Omega),$$

and thus

$$Z' = Z\left(1 + \frac{c^2 I}{2h\nu^3}\right),$$

where h is the Planck constant.

Another item of interest to consider here is the concept of local thermodynamics equilibrium (LTE), (see, e.g. [31,8,9]). To see the effect of the LTE assumption on Eq. (1.1), it is convenient to eliminate S, σ_a in (1.1) in favor of $\overline{B}$ and $a(\nu)$ defined by the relationships

$$S(\nu) = a(\nu)\overline{B}(\nu), \qquad \sigma_a(\nu) = a(\nu)\left(1 + \frac{c^2\overline{B}(\nu)}{2h\nu^3}\right),$$

and assume that $\sigma_s = 0$, where $\overline{B}$ is a function of ν only, and the absorption coefficient $a(\nu) = a(x, t, \nu, \Omega, \rho, \theta)$, we assume throughout this paper that $a(\nu) > 0$; more comments on $\overline{B}$ and $a(\nu)$ can be found in Remarks 2.2 and 3.1 of [28] as well as in [9]. Thus, from the "induced processes" and the LTE assumption together, $S(\nu)$, $\sigma_a(\nu)$ in (1.1) can be written as

$$\begin{aligned} S(\nu) &= a(\nu)\overline{B}(\nu)\left(1 + \frac{c^2 I(\nu, \Omega)}{2h\nu^3}\right), \\ \sigma_a(\nu) &= a(\nu)\left(1 + \frac{c^2\overline{B}(\nu)}{2h\nu^3}\right). \end{aligned} \tag{2.1}$$

Using (1.3), (2.1) and by the First Law of Thermodynamics $\theta dS = de + P_m d\tau$, where $\tau = 1/\rho$, we can rewrite Eqs. (1.1) and (1.2) as

$$\frac{1}{c}\frac{\partial I(\nu, \Omega)}{\partial t} + \Omega \cdot \nabla I(\nu, \Omega) = a(\nu)(\overline{B}(\nu) - I), \tag{2.2}$$

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla P_m + \nabla \cdot (\rho u \otimes u) = -\dfrac{1}{c}\displaystyle\int_0^\infty d\nu \int_{S^2} \Omega \left(a(\nu)(\overline{B}(\nu) - I)\right) d\Omega \\ S_t + u \cdot \nabla S = \dfrac{1}{c\rho\theta}\displaystyle\int_0^\infty d\nu \int_{S^2} u \cdot \Omega a(\nu)(\overline{B}(\nu) - I) d\Omega \\ \quad - \dfrac{1}{\rho\theta}\displaystyle\int_0^\infty d\nu \int_{S^2} a(\nu)(\overline{B}(\nu) - I) d\Omega, \end{cases} \tag{2.3}$$

where

$$P_m = A\exp\left(\frac{S}{c_\nu}\right)\rho^\gamma, \qquad \theta = A\frac{1}{c_\nu(\gamma - 1)}\exp\left(\frac{S}{c_\nu}\right)\rho^{\gamma - 1}, \tag{2.4}$$

with $A > 0$ constant, $c_\nu > 0$ being the heat conductivity and $\gamma > 1$ being the specific heat ratio.

We consider the Cauchy problem of (2.2)–(2.4) with the following initial data:

$$\begin{cases} I(x, 0, \nu, \Omega) = I^0(x, \nu, \Omega); \quad I^0(x, \nu, \Omega) = \overline{B}(\nu),\ |x| \geq R, \\ \rho(x, 0) = \rho^0(x) > 0; \quad \rho^0(x) = \overline{\rho},\ |x| \geq R, \\ u(x, 0) = u^0(x); \quad u^0(x) = \overline{u}(:= 0),\ |x| \geq R, \\ S(x, 0) = S^0(x); \quad S^0(x) = \overline{S},\ |x| \geq R, \end{cases} \tag{2.5}$$

where $R > 0$, $\overline{\rho} > 0$, $\overline{S}$ are some constant.

Let (ρ, u, S, I) be the local C^1 solution to the Cauchy problem (2.2)–(2.5) obtained in [28].

Lemma 2.1. (1) *If $I^0(x, \nu, \Omega) \geq \overline{B}(\nu)$ for $(x, \nu, \Omega) \in \mathbb{R}^3 \times \mathbb{R}^+ \times S^2$, then $I(x, t, \nu, \Omega) \geq \overline{B}(\nu)$ for $(x, \nu, \Omega) \in \mathbb{R}^3 \times \mathbb{R}^+ \times S^2$ and $t > 0$.*
(2) *If $I^0(x, \nu, \Omega) = \overline{B}(\nu)$ for $x \cdot \Omega < 0$ and $\nu \in \mathbb{R}^+$, then $I(x, t, \nu, \Omega) = \overline{B}(\nu)$ for $x \cdot \Omega < 0$, $\nu \in \mathbb{R}^+$, and $t > 0$.*

Proof. Since $\overline{B}$ is independent of x and t, Eq. (2.2) can be rewritten as

$$\frac{1}{c}\frac{\partial (I - \overline{B})}{\partial t} + \Omega \cdot \nabla (I - \overline{B}) + a(\nu)(I - \overline{B}) = 0.$$

An application of the method of characteristics to the above equation shows that $I(x, t, \nu, \Omega) - \overline{B}(\nu)$ has the same sign as $I^0(x - c\Omega t, \nu, \Omega) - \overline{B}(\nu)$.

(1) If $I^0(x,\nu,\Omega) \geq \overline{B}(\nu)$ for $(x,\nu,\Omega) \in \mathbb{R}^3 \times \mathbb{R}^+ \times S^2$, then $I^0(x - c\Omega t, \nu, \Omega) - \overline{B}(\nu) \geq 0$, thus $I(x,t,\nu,\Omega) - \overline{B}(\nu) \geq 0$, i.e., $I(x,t,\nu,\Omega) \geq \overline{B}(\nu)$.
(2) For $x \cdot \Omega < 0$, $I^0(x,\nu,\Omega) = \overline{B}(\nu)$, then $I^0(x - c\Omega t, \nu, \Omega) - \overline{B}(\nu) = 0$ since $(x - c\Omega t)\cdot \Omega = x\cdot\Omega - ct|\Omega|^2 < 0$, thus $I(x,t,\nu,\Omega) - \overline{B}(\nu) = 0$, i.e., $I(x,t,\nu,\Omega) = \overline{B}(\nu)$. □

The maximum speed of propagation of the front of a smooth disturbance is governed by the sound speed

$$\sigma = (\partial_\rho P_m(\overline{\rho}, \overline{S}))^{1/2} = \left(A\gamma\overline{\rho}^{\gamma-1}\exp\left(\frac{\overline{S}}{c_\nu}\right)\right)^{1/2},$$

since $\overline{u} = 0$. More precisely, letting

$$D(t) = \{x \in \mathbb{R}^3 : |x| \geq R + \sigma t\},$$

we have the following property:

Proposition 2.1 (*Finite Propagation Speed*)**.** *If (I,ρ,u,S) is a C^1 solution of (2.2)–(2.5), then $(I,\rho,u,S) \equiv (\overline{B}(\nu), \overline{\rho}, 0, \overline{S})$ on $D(t)$, $0 \leq t \leq T$ for any fixed $T > 0$.*

The proof of this proposition in the same manner as in [28, Proposition 3.1].

Our main result is Theorem 2.1 below, which establishes the finite-time formation of singularities without any condition of largeness on initial data (2.5). Let us define the functions

$$q^0(r) = \int_{|x|>r} |x|^{-1}(|x| - r)^2(\rho^0(x) - \overline{\rho})dx,$$
$$q^1(r) = \int_{|x|>r} |x|^{-3}(|x|^2 - r^2)x \cdot \rho^0(x)u^0(x)dx.$$

Theorem 2.1 (*Main Result*)**.** *Let (I,ρ,u,S) be a C^1 solution of (2.2)–(2.5), suppose that for some R_0 with $0 < R_0 < R$,*

$$q^0(r) > 0, \qquad q^1(r) \geq 0, \tag{2.6}$$

for $R_0 < r < R$; also suppose

$$S^0(x) \geq \overline{S}; \tag{2.7}$$

and, for $(x,\nu,\Omega) \in \mathbb{R}^3 \times \mathbb{R}^+ \times S^2$,

$$I^0(x,\nu,\Omega) \geq \overline{B}(\nu), \quad \textit{and in addition}, \quad I^0(x,\nu,\Omega) = \overline{B}(\nu) \quad \textit{if } x\cdot\Omega < 0. \tag{2.8}$$

Then the life span T is finite.

In what follows, all generic constants will be denoted by C which may depend on the fixed constants R and R_0, but is independent of the initial data.

3. Proof of the main result: Theorem 2.1

In this section, we prove our main result in Theorem 2.1. For simplicity, we first consider the case $\gamma = 2$, and later we will indicate what modifications are necessary for the general case. The proof will be divided into two steps as in the two lemmas below.

For a C^1 solution (I,ρ,u,S), we know that $\rho - \overline{\rho}$ is supported in $B(t) = \{x \in \mathbb{R}^3 : |x| \leq R + \sigma t\}$ by Proposition 2.1. So, we can define

$$H(r,t) = \int_{|x|>r} \omega(x,r)\,(\rho(x,t) - \overline{\rho})\,dx, \quad r > 0,$$

where

$$\omega(x,r) = |x|^{-1}(|x| - r)^2.$$

Then, we have following inequalities.

Lemma 3.1.

$$H(r,t) \geq H^0(r,t) + \frac{1}{2\sigma}\int_0^t \int_{r-\sigma(t-\tau)}^{r+\sigma(t-\tau)} G(y,\tau)dyd\tau, \tag{3.1}$$

$$G(r,t) \geq 0, \tag{3.2}$$

where

$$H^0(r,t) = \frac{1}{2}\left(q^0(r+\sigma t) + q^0(r-\sigma t) + \frac{1}{\sigma}\int_{r-\sigma t}^{r+\sigma t} q^1(y)dy\right), \tag{3.3}$$

$$G(r,t) = \int_{|x|>r} 2|x|^{-1}\left(P_m - \overline{P}_m - \sigma^2(\rho - \overline{\rho})\right)dx. \tag{3.4}$$

Proof. Using the first equation of (2.3) and integration by parts,

$$\begin{aligned}\frac{\partial}{\partial t}H(r,t) &= \int_{|x|>r} \omega(x,r)\frac{\partial}{\partial t}(\rho(x,t) - \overline{\rho})dx \\ &= -\int_{|x|>r} \omega(x,r)\nabla\cdot\rho u(x,t)dx \\ &= \int_{|x|>r} \nabla\omega(x,r)\cdot\rho u(x,t)dx,\end{aligned}$$

since ρu is supported in $B(t)$ and $\omega(x,r) = 0$ when $|x| = r$. Thus, $H(r,t)$ is C^2 in t, and we can differentiate it again using (2.3)

$$\begin{aligned}\frac{\partial^2}{\partial t^2}H(r,t) &= \int_{|x|>r} \nabla\omega(x,r)\cdot\frac{\partial}{\partial t}(\rho u)(x,t)dx \\ &= -\sum_{i,j}\int_{|x|>r}\frac{\partial\omega}{\partial x_i}\frac{\partial}{\partial x_j}(\rho u_i u_j)dx - \int_{|x|>r}\nabla\omega\cdot\nabla(P_m - \overline{P}_m)dx \\ &\quad + \frac{1}{c}\int_{|x|>r}\int_0^\infty d\nu\int_{S^2}\nabla\omega\cdot\Omega(a(\nu)(I - \overline{B}))d\Omega dx,\end{aligned}$$

where $\overline{P}_m = P_m(\overline{\rho}, \overline{S})$. Now since

$$\nabla\omega(x,r) = |x|^{-3}(|x|^2 - r^2)x,$$

which vanishes on $\{x : \ |x| = r\}$, and since $\rho u_i u_j$ and $P_m - \overline{P}_m$ have compact support, we integrate by parts again to obtain

$$\begin{aligned}\frac{\partial^2}{\partial t^2}H(r,t) &= \sum_{i,j}\int_{|x|>r}\frac{\partial^2\omega}{\partial x_i\partial x_j}\rho u_i u_j dx + \int_{|x|>r}\triangle\omega\cdot(P_m - \overline{P}_m)dx \\ &\quad + \frac{1}{c}\int_{|x|>r}\int_0^\infty d\nu\int_{S^2}\nabla\omega\cdot\Omega(a(\nu)(I - \overline{B}))d\Omega dx \\ &\equiv I_1(r,t) + I_2(r,t) + I_3(r,t).\end{aligned} \tag{3.5}$$

A simple computation of $\frac{\partial^2\omega}{\partial x_i\partial x_j}$ shows that

$$\begin{aligned}I_1(r,t) &= \int_{|x|>r}\frac{2r^2}{|x|^3}\cdot\rho\cdot\left(\frac{x}{|x|}\cdot u\right)^2 dx - \int_{|x|>r}\frac{|x|^2 - r^2}{|x|^3}\cdot\rho\cdot\left(\frac{x}{|x|}\cdot u\right)^2 dx \\ &\quad + \int_{|x|>r}\frac{|x|^2 - r^2}{|x|^3}\cdot\rho\cdot|u|^2 dx \geq 0,\end{aligned} \tag{3.6}$$

since $\left(\frac{x}{|x|}\cdot u\right)^2 \leq |u|^2$.

Thanks to (2.8) and Lemma 2.1, we can deduce that

$$I_3(r,t) = \frac{1}{c}\int_{|x|>r}\int_0^\infty d\nu\int_{S^2}|x|^{-3}(|x|^2 - r^2)x\cdot\Omega(a(\nu)(I - \overline{B}))d\Omega dx \geq 0.$$

For the second term I_2, since

$$\triangle\omega(x,r) = 2|x|^{-1} = \omega_{rr}(x,r),$$

thus,

$$I_2(r,t) = \int_{|x|>r} 2|x|^{-1}(P_m - \overline{P}_m)dx = \frac{\partial^2}{\partial r^2}\int_{|x|>r}\omega(x,r)(P_m - \overline{P}_m)dx, \tag{3.7}$$

because ω and ω_r vanish on $\{x: \ |x| = r\}$. Combination of (3.5)–(3.7) gives

$$\left(\frac{\partial^2}{\partial t^2} - \sigma^2\frac{\partial^2}{\partial r^2}\right)H(r,t) \geq G(r,t), \tag{3.8}$$

where

$$\begin{aligned} G(r,t) \equiv \frac{\partial^2}{\partial r^2}\widetilde{G}(r,t) &\equiv \frac{\partial^2}{\partial r^2}\int_{|x|>r}\omega(x,r)\left(P_m - \overline{P}_m - \sigma^2(\rho-\overline{\rho})\right)dx \\ &= G(r,t) = \int_{|x|>r} 2|x|^{-1}\left(P_m - \overline{P}_m - \sigma^2(\rho - \overline{\rho})\right)dx. \end{aligned} \tag{3.9}$$

Inversion of the one dimensional d'Alembertian $\square = \frac{\partial^2}{\partial t^2} - \sigma^2\frac{\partial^2}{\partial r^2}$ in (3.8) for $r > R_0 + \sigma t$ yields

$$\begin{aligned} H(r,t) &= H^0(r,t) + \frac{1}{2\sigma}\int_0^t\int_{r-\sigma(t-\tau)}^{r+\sigma(t-\tau)}\square H(y,\tau)dyd\tau \\ &\geq H^0(r,t) + \frac{1}{2\sigma}\int_0^t\int_{r-\sigma(t-\tau)}^{r+\sigma(t-\tau)}G(y,\tau)dyd\tau. \end{aligned}$$

We note that as long as u is C^1, the particle paths

$$\frac{dx}{dt} = u(x,t); \quad x(0,\xi) = \xi$$

exist. Using the third equation of (2.3), (2.8) and Lemma 2.1, we can deduce that

$$\begin{aligned} \frac{dS(x,t)}{dt} &= S_t + u\cdot\nabla S \\ &= \frac{1}{c\rho\theta}\int_0^\infty d\nu\int_{S^2} u\cdot\Omega a(\nu)(\overline{B}(\nu) - I)d\Omega - \frac{1}{\rho\theta}\int_0^\infty d\nu\int_{S^2} a(\nu)(\overline{B}(\nu) - I)d\Omega \\ &\geq \frac{1}{c\rho\theta}\int_0^\infty d\nu\int_{S^2}|u||\Omega|a(\nu)(\overline{B}(\nu) - I)d\Omega - \frac{1}{\rho\theta}\int_0^\infty d\nu\int_{S^2} a(\nu)(\overline{B}(\nu) - I)d\Omega \\ &= \frac{1}{c\rho\theta}\int_0^\infty d\nu\int_{S^2}(|u||\Omega| - c)a(\nu)(\overline{B}(\nu) - I)d\Omega \\ &\geq 0, \end{aligned}$$

since $\Omega \in S^2$, $|\Omega| = 1$ and $|u| \ll c$. So by (2.7) we have that

$$S(x,t) \geq S^0 \geq \overline{S}.$$

Consequently, $P_m(\rho,S) \geq P_m(\rho,\overline{S})$, so that

$$P_m - \overline{P}_m - \sigma^2(\rho - \overline{\rho}) \geq A\exp\left(\frac{S}{c_\nu}\right)(\rho^2 - \overline{\rho}^2 - 2\overline{\rho}(\rho - \overline{\rho})) = A\exp\left(\frac{S}{c_\nu}\right)(\rho - \overline{\rho})^2. \tag{3.10}$$

It follows from (3.4) that

$$G(r,t) \geq 0.$$

The proof of Lemma 3.1 is complete. $\square$

Now, define the C^2-function

$$F(t) = \int_0^t (t-\tau)\int_{\sigma\tau+R_0}^{\sigma\tau+R} r^{-1}H(r,\tau)drd\tau,$$

then, we have following estimates.

Lemma 3.2.

$$F''(t) \geq C\frac{\sigma^4}{\overline{\rho}}\left((\sigma t+R)^3 \ln\left(\frac{\sigma t+R}{R}\right)\right)^{-1} F(t)^2, \quad t \geq R_1,$$
$$F''(t) \geq B_0(\sigma t+R)^{-1}, \quad t>0,$$
$$F'(t) \geq \sigma^{-1}B_0 \ln\left(\frac{\sigma t+R}{R}\right), \quad t>0,$$
$$F(t) \geq C\sigma^{-2}B_0(\sigma t+R)\ln\left(\frac{\sigma t+R}{R}\right), \quad t>R_1,$$

where

$$R_1 = \frac{1}{2\sigma}(R-R_0), \qquad B_0 = \frac{1}{2}\int_{R_0}^{R} q^0(r)dr.$$

Proof. Since $F(t)$ is C^2, we have, from (3.1),

$$\begin{aligned} F''(t) &= \int_{\sigma t+R_0}^{\sigma t+R} r^{-1}H(r,t)dr \\ &\geq \int_{\sigma t+R_0}^{\sigma t+R} r^{-1}H^0(r,t)dr + \frac{1}{2\sigma}\int_{\sigma t+R_0}^{\sigma t+R} r^{-1}\int_0^t \int_{r-\sigma(t-\tau)}^{r+\sigma(t-\tau)} G(y,\tau)dyd\tau dr \\ &\equiv J_1+J_2. \end{aligned} \tag{3.11}$$

By our hypotheses (2.6), $q^0(r)>0$ and $q^1(r)\geq 0$ on $R_0<r<R$. Hence, we see from (3.3) that

$$J_1 \geq \frac{1}{2}\int_{\sigma t+R_0}^{\sigma t+R} r^{-1}q^0(r-\sigma t)dr \geq \frac{1}{2}(\sigma t+R)^{-1}\int_{\sigma t+R_0}^{\sigma t+R} q^0(r-\sigma t)dr = B_0(\sigma t+R)^{-1}>0. \tag{3.12}$$

In order to estimate J_2 from below, we write it as

$$\begin{aligned} J_2 &= \frac{1}{2\sigma}\int_0^{t-R_1}\int_{\sigma\tau+R_0}^{\sigma\tau+R} G(y,\tau)\int_{\sigma t+R_0}^{y+\sigma(t-\tau)} r^{-1}drdyd\tau + \frac{1}{2\sigma}\int_{t-R_1}^{t}\int_{\sigma\tau+R_0}^{2\sigma t-\sigma\tau+R_0} G(y,\tau)\int_{\sigma t+R_0}^{y+\sigma(t-\tau)} r^{-1}drdyd\tau \\ &\quad + \frac{1}{2\sigma}\int_{t-R_1}^{t}\int_{2\sigma t-\sigma\tau+R_0}^{\sigma\tau+R} G(y,\tau)\int_{y-\sigma(t-\tau)}^{y+\sigma(t-\tau)} r^{-1}drdyd\tau \\ &\equiv J_2^1+J_2^2+J_2^3. \end{aligned}$$

In J_2^1 we have

$$\begin{aligned} \int_{\sigma t+R_0}^{y+\sigma(t-\tau)} r^{-1}dr &\geq (y+\sigma(t-\tau))^{-1}(y-\sigma\tau-R_0) \\ &\geq (\sigma t+R)^{-1}(y-\sigma\tau-R_0) \\ &\geq C(\sigma t+R)^{-1}\left(\frac{t-\tau}{t}\right)(y-\sigma\tau-R_0)^2 \\ &\geq C\sigma(\sigma t+R)^{-2}(t-\tau)(y-\sigma\tau-R_0)^2. \end{aligned}$$

In J_2^2 we have

$$\begin{aligned} \int_{\sigma t+R_0}^{y+\sigma(t-\tau)} r^{-1}dr &\geq C(\sigma t+R)^{-1}(y-\sigma\tau-R_0) \\ &\geq C\sigma(\sigma t+R)^{-2}(t-\tau)(y-\sigma\tau-R_0)^2 \end{aligned}$$

and in J_2^3 we have

$$\begin{aligned} \int_{y-\sigma(t-\tau)}^{y+\sigma(t-\tau)} r^{-1}dr &\geq 2\sigma t(y+\sigma(t-\tau)) \\ &\geq C\sigma(\sigma t+R)^{-1}(t-\tau) \\ &\geq C\sigma(\sigma t+R)^{-2}(t-\tau)(y-\sigma\tau-R_0)^2. \end{aligned}$$

From the above estimates and using (3.2), it follows that

$$J_2 \geq C(\sigma t + R)^{-2} \int_0^t \int_{\sigma\tau+R_0}^{\sigma\tau+R} (t-\tau)(y-\sigma\tau-R_0)^2 G(y,\tau)dyd\tau,$$

for $t \geq R_1$. Returning to (3.9), and noting that $\widetilde{G}(y,\tau)$ vanishes for $y > \sigma\tau + R$, we can integrate by parts to obtain

$$\begin{aligned} J_2 &\geq C(\sigma t + R)^{-2} \int_0^t \int_{\sigma\tau+R_0}^{\sigma\tau+R} (t-\tau)\widetilde{G}(y,\tau)dyd\tau \\ &\geq C(\sigma t + R)^{-2} \frac{\sigma^2}{\overline{\rho}} \int_0^t (t-\tau) \int_{\sigma\tau+R_0}^{\sigma\tau+R} \int_{|x|>y} \omega(\rho-\overline{\rho})^2 dxdyd\tau \\ &\equiv C(\sigma t + R)^{-2} \frac{\sigma^2}{\overline{\rho}} J_3. \end{aligned}$$

However, Schwarz's inequality yields

$$\begin{aligned} F^2(t) &\leq J_3 \left(\int_0^t (t-\tau) \int_{\sigma\tau+R_0}^{\sigma\tau+R} y^{-2} \int_{y<|x|<\sigma\tau+R} \omega(x,y)dxdyd\tau \right) \\ &\equiv J_3 J_4. \end{aligned} \tag{3.13}$$

Thus,

$$J_2 \geq C(\sigma t + R)^{-2} \frac{\sigma^2}{\overline{\rho}} J_4^{-1} F^2(t). \tag{3.14}$$

We estimate J_4 as follows:

$$\begin{aligned} J_4 &= \int_0^t (t-\tau) \int_{\sigma\tau+R_0}^{\sigma\tau+R} y^{-2} 4\pi \int_y^{\sigma\tau+R} |x|(|x|-y)^2 d|x|dyd\tau \\ &\leq C \int_0^t (t-\tau) \int_{\sigma\tau+R_0}^{\sigma\tau+R} y^{-2}(\sigma\tau+R)(\sigma\tau+R-y)^3 dyd\tau \\ &\leq C \int_0^t (t-\tau)(\sigma\tau+R) \int_{\sigma\tau+R_0}^{\sigma\tau+R} y^{-2} dyd\tau \\ &\leq C\sigma^{-2}(\sigma\tau+R) \ln\left(\frac{\sigma t+R}{R}\right). \end{aligned} \tag{3.15}$$

Now, the combination of (3.11), (3.14) and (3.15) yields the inequality

$$F''(t) \geq C \frac{\sigma^4}{\overline{\rho}} \left((\sigma t+R)^3 \ln\left(\frac{\sigma t+R}{R}\right) \right)^{-1} F^2(t), \quad t \geq R_1,$$

since $J_1 > 0$. On the other hand, since $J_2 \geq 0$, (3.11) and (3.12) yield the estimates:

$$\begin{aligned} &F''(t) \geq B_0(\sigma t+R)^{-1}, \quad t > 0, \\ &F'(t) \geq \sigma^{-1} B_0 \ln\left(\frac{\sigma t+R}{R}\right), \quad t > 0, \\ &F(t) \geq C\sigma^{-2} B_0(\sigma t+R) \ln\left(\frac{\sigma t+R}{R}\right), \quad t > R_1, \end{aligned}$$

since $F(0) = F'(0) = 0$. The proof of Lemma 3.2 is complete. □

Now we can use Lemma 3.2 to prove the finite-time blowup of C^1 solutions. Indeed, if we let $s = \sigma t$ and $\widetilde{F}(s) = \frac{\sigma^2}{\overline{\rho}} F(t)$, then we have $\widetilde{F}(0) = \widetilde{F}'(0) = 0$, and

$$\begin{aligned} &\widetilde{F}''(s) \geq C \left((s+R)^3 \ln\left(\frac{s+R}{s}\right) \right)^{-1} \widetilde{F}^2(s), \quad s > \frac{1}{2}(R-R_0) = k_1, \\ &\widetilde{F}'(s) \geq C \frac{B_0}{\overline{\rho}} \ln\left(\frac{s+R}{s}\right), \qquad \widetilde{F}''(s) \geq \frac{B_0}{\overline{\rho}}(s+R)^{-1}, \quad s > 0, \\ &\widetilde{F}(s) \geq C \frac{B_0}{\overline{\rho}} (s+R) \ln\left(\frac{s+R}{s}\right), \quad s > k_1. \end{aligned}$$

In [27], we showed that any C^2 functions $\widetilde{F}(s)$ which satisfies the above inequalities has a finite life span bounded above by $\frac{C}{\sigma}\exp(C\sigma^2/B_0^2)$. (cf. The conclusion of the proof of Lemma 3.3 in [27].) Hence, the life span of the C^1 solution is bounded above by

$$\frac{C}{\sigma}\exp(C\sigma^2/B_0^2).$$

We finally turn our attention to the general case, $\gamma > 1$, the adjustment is needed in (3.10) which now becomes

$$P_m - \overline{P}_m - \sigma^2(\rho - \overline{\rho}) \geq A\exp\left(\frac{S}{c_v}\right)\left(\rho^\gamma - \overline{\rho}^\gamma - \gamma\overline{\rho}^{\gamma-1}(\rho - \overline{\rho})\right) \equiv A\exp\left(\frac{S}{c_v}\right)\Psi(\rho, \overline{\rho}).$$

Since ρ^γ is convex, then

$$\Psi(\rho, \overline{\rho}) = \rho^\gamma - \overline{\rho}^\gamma - \gamma\overline{\rho}^{\gamma-1}(\rho - \overline{\rho}) > 0, \quad \text{for } \rho \neq \overline{\rho}.$$

However, by Taylor's theorem one has

$$\Psi(\rho, \overline{\rho}) \geq C_0(\gamma, \overline{\rho})(\rho - \overline{\rho})^2, \quad \text{for } 0 < \rho < 2\overline{\rho},$$

and some positive constant $C_0(\gamma, \overline{\rho})$. On the other hand, there exists a positive constant $C_1(\gamma)$ such that

$$\Psi(\rho, \overline{\rho}) \geq C_1(\gamma)(\rho - \overline{\rho})^\gamma \quad \text{for } \rho \geq 2\overline{\rho}.$$

Therefore, there exists a positive constant $C(\gamma, \overline{\rho})$ such that

$$\Psi(\rho, \overline{\rho}) \geq C(\gamma, \overline{\rho})\Phi_\gamma(\rho - \overline{\rho}),$$

where Φ_γ is a nonnegative convex function given by

$$\Phi_\gamma(\rho - \overline{\rho}) = \begin{cases} (\rho - \overline{\rho})^2, & 0 < \rho < 2\overline{\rho}, \\ (\rho - \overline{\rho})^\gamma, & \rho \geq 2\overline{\rho}. \end{cases}$$

Finally, Jensen's inequality should be used instead of Schwarz's inequality in (3.13). The rest of the details should then follow accordingly, and the resulting differential inequality still has a finite life span. However, the upper bound for the local existence time T will be different from the one which we have obtained for the case $\gamma = 2$. The proof of Theorem 2.1 is complete.

Remark 3.1. The conclusion in Theorem 2.1 is also hold for the two-dimensional case. In order to prove this result, we should modify functions q^0, q^1. Let

$$q^0(r) = \int_{x_1 > r} w(x, r)(\rho^0(x) - \overline{\rho})dx,$$

$$q^1(r) = \int_{x_1 > r} w(x, r)\rho^0(x)u_1^0(x)dx,$$

where $w(x, r) = (x_1 - r)^2$, $u_1^0(x)$ is the first component of $u^0(x) = (u_1^0(x), u_2^0(x))$. The proof of this result is similar to Theorem 2.1, so we omit here.

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